SOLUTION OF THE NONLINEAR INVERSE PROBLEM FOR A GENERALIZED HEAT-CONDUCTION EQUATION IN A REGION WITH MOVING BOUNDARIES

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In an analysis of the numerical solution of the nonlinear inverse heat-conduction problem in a region with moving boundaries, a regularization method is used to construct an algorithm for smoothing the experimental data in a compilation of the input data for the inverse problem.

The inverse heat-conduction problems constitute an extremely important and rapidly developing branch of the theory of the unsteady thermal experiment. Of particular importance in studies of high-temperature processes are the nonlinear formulations of the inverse problems, in which the thermal properties of the material depend on the temperature.

Below we outline a numerical method for solving the inverse problem for the case of the one-dimensional quasilinear heat-conduction equation with a continuous heat source and a convective term. This model corresponds, in particular, to the three-dimensional thermal destruction of the material of an object, with the flow of the resulting gaseous products in the pores of the object.

This method is based on an implicit difference scheme; the integration is carried out along the direction of the spatial variable [1]. This approach to the solution of this problem is taken under the assumption that implicit difference schemes should have significant "viscous" properties for this choice of the direction of integration of the heat-conduction equation. This hypothesis is verified in the numerical use of the method, so that it becomes possible to obtain a regular solution of the inverse problem with short time steps, even if the input temperatures are afflicted with certain (small) errors.

The analytic form of this problem is as follows:

$$C(T)\frac{\partial T}{\partial t}=\frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right)+k(T)\frac{\partial T}{\partial x}+\psi(T), \tag{1}$$

$$X_1(t)<x<X_3(t),\quad 0<t\leqslant t_m,$$

$$T(x,\ 0)=\varphi(x),\ X_1(0)\leqslant x\leqslant X_3(0), \tag{2}$$

$$-\lambda(T(X_3(t),\ t))\frac{\partial T(X_3(t),\ t)}{\partial x}=q_3(t), \tag{3}$$

$$T(X_2(t),\ t)=f(t),\ X_1(t)<X_2(t)\leqslant X_3(t), \tag{4}$$

where $C(T)$, $\lambda(T)$, $k(T)$, $\psi(T)$, $X_1(t)$, $X_2(t)$, $q_3(t)$, and $f(T)$ are known functions. We are to determine the heat flux at the left-hand boundary of the region and the temperature field within the object under initial condition (2) and under the boundary condition at the right-hand boundary, condition (3). We also know the temperature at a point within the object. The problem is complicated by the external moving boundaries and the moving boundary with the known temperature. From the physical standpoint, the motion

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of these boundaries, which is described by the functions $X_1(t)$, $X_2(t)$, and $X_3(t)$, can correspond to linear removal, thermal shrinkage, or expansion of the material.

The first step in the solution of problem (1)-(4) is to reduce it to a Cauchy problem. For this purpose we need to find the temperature field in the region $D_2\{X_2(t) \le x \le X_3(t),\ 0 \le t \le t_m\}$ and thus to determine the heat flux $q_2(t)$ at the boundary $X_2(t)$. This is a well-studied boundary-value problem, and numerical solution methods are available [2, 3].

In the region $D_1\{X_1(t) \le x \le X_2(t),\ 0 \le t \le t_m\}$ we are thus confronted with the problem of determining the temperature field and the boundary conditions $T[X_1(t), t]$, $q_1(t)$ from the known functions $T[X_2(t), t]$ and $q_2(t)$:

$$C(T)\frac{\partial T}{\partial t} = \frac{\partial}{\partial x}\left(\lambda(T)\frac{\partial T}{\partial x}\right) + k(T)\frac{\partial T}{\partial x} + \psi(T), \tag{5}$$

$$X_1(t) < x < X_2(t),\ 0 < t \leqslant t_m,$$

$$T(x, 0) = \varphi(x),\ X_1(0) \leqslant x \leqslant X_2(0), \tag{6}$$

$$-\lambda(T(X_2(t), t))\frac{\partial T(X_2(t), t)}{\partial x} = q_2(t), \tag{7}$$

$$T(X_2(t), t) = f(t). \tag{8}$$

From the computational standpoint it is more convenient to use a rectangular range for the independent variables. To carry out the appropriate transformation of the original region, we use the new variables [4]

$$\xi = \frac{x - X_1(t)}{X_2(t) - X_1(t)},\ \tau = t, \tag{9}$$

which "straighten out" the fronts. Then (5)-(8) can be rewritten as

$$C(T)\frac{\partial T}{\partial \tau} = \frac{1}{[X_2(\tau) - X_1(\tau)]^2}\left[\lambda(T)\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial \lambda}{\partial T}\left(\frac{\partial T}{\partial \xi}\right)^2\right] + \frac{C(T)[\dot{X}_1(\tau) + \xi(\dot{X}_2(\tau) - \dot{X}_1(\tau))] + k(T)}{X_2(\tau) - X_1(\tau)}\frac{\partial T}{\partial \xi} + \psi(T), \tag{10}$$

$$0 < \xi < 1,\ 0 < \tau \leqslant \tau_m,$$

$$T(\xi, 0) = \varphi(\xi),\ 0 \leqslant \xi \leqslant 1, \tag{11}$$

$$-\frac{\lambda(T(1, \tau))}{X_2(\tau) - X_1(\tau)} \cdot \frac{\partial T(1, \tau)}{\partial \xi} = q_2(\tau), \tag{12}$$

$$T(1, \tau) = f(\tau). \tag{13}$$

We introduce the difference grid

$$h,\ \Delta\tau\,(\xi_i = hi,\ i = 0, 1, \ldots, n;\ \tau_j = \Delta\tau \cdot j,\ j = 0, 1, \ldots, m),$$

where h is the spatial step and $\Delta\tau$ is the time step. Approximating the derivatives by

$$\left(\frac{\partial T}{\partial \tau}\right)_i^j = \frac{T_i^{j+1} - T_i^{j-1}}{2\Delta\tau},\quad \left(\frac{\partial T}{\partial \xi}\right)_i^j = \frac{T_{i+1}^j - T_i^j}{h},$$

$$\left(\frac{\partial^2 T}{\partial \xi^2}\right)_i^j = \frac{T_{i+2}^j - 2T_{i+1}^j + T_i^j}{h^2},$$

we find the following difference analog of Eq. (10) for each i-th spatial layer:

$$A_i^j T_i^{j+1} + B_i^j T_i^j + D_i^j T_i^{j-1} = F_i^j,\ j = 1, 2, \ldots, m-1, \tag{14}$$

where

$$A_i^j = \frac{C_i^j}{2\Delta\tau},$$

$$B_i^j = \frac{C_i^j[\dot{X}_1^j + \xi_i(\dot{X}_2^j - \dot{X}_1^j)] + k_i^j}{h(X_2^j - X_1^j)} - \frac{\lambda_i^j}{h^2(X_2^j - X_1^j)^2},$$

$$D_i^j = -\frac{C_i^j}{2\Delta\tau},$$

$$F_i^j = \frac{1}{(X_2^j - X_1^j)^2}\left[\lambda_i^j \frac{T_{i+2}^j + 2T_{i+1}^j}{h^2} + \left(\frac{\partial\lambda}{\partial T}\right)_i^j \left(\frac{T_{i+1}^j - T_i^j}{h}\right)^2\right] + \frac{C_i^j [\dot{X}_1^j + \xi_i (\dot{X}_2^j - \dot{X}_1^j)] + k_i^j}{h (X_2^j - X_1^j)} T_{i+1}^j + \psi_i^j.$$

To close the system of equations we also need to specify two boundary conditions, at $\tau = 0$ and $\tau = \tau_m$, where τ_m is the right-hand limit of the time interval. From the initial temperature distribution we have $T_i^0 = \varphi_i$. As the second condition we can use one of the a priori relations

$$\frac{\partial T_i(\tau_m)}{\partial\tau} \approx \frac{T_i^m - T_i^{m-1}}{\Delta\tau} = b_1 \equiv \frac{df(\tau_m)}{d\tau} = \text{const}, \tag{15}$$

$$\frac{\partial^2 T_i(\tau_m)}{\partial\tau^2} \approx \frac{T_i^{m+1} - 2T_i^m + T_i^{m-1}}{\Delta\tau^2} = b_2 \equiv \frac{d^2 f(\tau_m)}{d\tau^2} = \text{const}, \quad i = n-1,\ n-2,\ \ldots,\ 0 \tag{16}$$

or we can approximate Eq. (10) with $j = m$ by the difference scheme

$$\frac{T_i^m - T_i^{m-1}}{\Delta\tau} = M_i^m, \tag{17}$$

where M_i^m is an operator approximating the right side of Eq. (10). Conditions (16) and (17) cause smaller distortions in the vicinity of the right-hand limit of the time interval.

In relation (16) the quantity T_i^{m+1} can be taken from difference equation (10) with $j = m$.

Accordingly, in each i-th spatial layer we must solve a system of nonlinear algebraic equations [in the case $\tau = \tau_m$ we can use, e.g., relation (15)]:

$$\begin{gathered} T_i^0 = \varphi_i, \\ A_i^j T_i^{j+1} + B_i^j T_i^j + D_i^j T_i^{j-1} = F_i^j,\ j = 1,\ 2,\ \ldots,\ m-1, \\ T_i^m = T_i^{m-1} + b_1\Delta\tau. \end{gathered} \tag{18}$$

System (18) is solved by the pivotal condensation method with iterations in terms of the coefficients. The iterative process is ended when the condition

$$\max_j |T_i^{j(p+1)} - T_i^{j(p)}| \leqslant \varepsilon$$

becomes satisfied; here p is the number of the iterative step and $\varepsilon > 0$ is a prespecified quantity.

Solving problem (18) in each spatial layer, we find the temperature field in region D_1. Then the heat flux can be determined by a simple calculation on the basis of a finite-difference approximation of the boundary condition at the left-hand boundary:

$$-\frac{\lambda(T(0,\ \tau))}{X_2(\tau) - X_1(\tau)} \cdot \frac{\partial T(0,\ \tau)}{\partial\xi} = q_1(\tau). \tag{19}$$

To begin the calculation process, we need to specify the temperature profiles at the first two spatial layers. The first profile is determined from condition (13), while the second is calculated through a finite-difference approximation of boundary condition (7).

This algorithm is an instance of one of the direct methods for solving the nonlinear inverse heat-conduction problem with Cauchy data. Because of the "incorrect" nature of the initial formulation of the problem, the calculation becomes unstable under certain conditions, and a regular solution can be found by appropriately choosing the time of the integration, $\Delta\tau$ [1].

Errors in the input data, which are always present, since measuring instruments are imperfect, strongly influence the degree of instability. When direct methods are used for solving the inverse problems it is possible to reduce that minimum value of the integration time step for which the solution is still smooth, by smoothing the input temperatures. The treatment of the input data should lead to a uniform approximation of not only the function itself but also its first derivatives.

In general, a problem of this type is an "incorrectly" formulated problem, and it can be thought of as the problem of using a regularization method to solve the operator equation

$$Au = f_\delta, \tag{20}$$

where f_δ are the input data, specified with some errors, and A is the unit operator.

We assume that the function f_δ is given on the uniform grid $\tau_j = \Delta\tau \cdot j$, $j = 0, 1, 2, \ldots, m$.

We write the regularizing functional of A. N. Tikhonov in the form

$$\Phi(\alpha) = \|Au - f_\delta\|^2 + \alpha k_1 \left\| \frac{du}{d\tau} - \frac{du^*}{d\tau} \right\|^2 + \alpha k_2 \left\| \frac{d^2u}{d\tau^2} - \frac{d^2u^*}{d\tau^2} \right\|^2$$

or

$$\Phi(\alpha) = \sum_{j=1}^{m} (u_j - f_j)^2 + \frac{\alpha k_1}{\Delta\tau^2} \sum_{j=1}^{m} [(u_j - u_{j-1}) - (u_j^* - u_{j-1}^*)]^2 + \frac{\alpha k_2}{\Delta\tau^4} \sum_{j=1}^{m} [(u_{j+1} - 2u_j + u_{j-1}) - (u_{j+1}^* - 2u_j^* + u_{j-1}^*)]^2, \tag{21}$$

where u_j^*, $j = 0, 1, \ldots, m+1$ serves as the "trial" solution [6], and k_1 and k_2 are certain nonnegative numbers.

Minimizing (21) with respect to all u_j, $j = 1, 2, \ldots, m$, we find a system of linear algebraic equations with a symmetric five-diagonal matrix for determining the regularized solution:

$$\sum_{k=j-2}^{j=2} a_{j,k} u_j = b_j, \quad j = 1, 2, \ldots, m, \tag{22}$$

where

$$a_{k,k} = \begin{cases} 1 + 2\dfrac{\alpha k_1}{\Delta\tau^2} + 5\dfrac{\alpha k_2}{\Delta\tau^4}, & k = 1, m-1, \\ 1 + 2\dfrac{\alpha k_1}{\Delta\tau^2} + 6\dfrac{\alpha k_2}{\Delta\tau^4}, & k = 2, 3, \ldots, m-2, \\ 1 + \dfrac{\alpha k_1}{\Delta\tau^2} + \dfrac{\alpha k_2}{\Delta\tau^4}, & k = m; \end{cases}$$

$$a_{k,k-1} = a_{k-1,k} = \begin{cases} -\dfrac{\alpha k_1}{\Delta\tau^2} - 4\dfrac{\alpha k_3}{\Delta\tau^4}, & k = 2, 3, \ldots, m-1, \\ -\dfrac{\alpha k_1}{\Delta\tau^2} - 2\dfrac{\alpha k_2}{\Delta\tau^4}, & k = m, \\ 0, & k = 1; \end{cases}$$

$$a_{k,k-2} = a_{k-2,k} = \begin{cases} 0, & k = 1, 2, \\ \dfrac{\alpha k_2}{\Delta\tau^4}, & k = 3, 4, \ldots, m; \end{cases}$$

$$b_1 = f_1 - \frac{\alpha k_1}{\Delta\tau^2}(u_2^* - 2u_1^* + u_0^* + u_0) + \frac{\alpha k_2}{\Delta\tau^4}(u_3^* - 4u_2^* + 5u_1^* - 2u_0^* + 2u_0);$$

$$b_2 = f_2 - \frac{\alpha k_1}{\Delta\tau^2}(u_3^* - 2u_2^* + u_1^*) + \frac{\alpha k_2}{\Delta\tau^4}(u_4^* - 4u_3^* + 6u_2^* - 4u_1^* + u_0^* - u_0);$$

$$b_k = f_k - \frac{\alpha k_1}{\Delta\tau^2}(u_{k+1}^* - 2u_k^* + u_{k-1}^*) + \frac{\alpha k_2}{\Delta\tau^4}(u_{k+2}^* - 4u_{k+1}^* + 6u_k^* - 4u_{k-1}^* + u_{k-2}^*);$$

$$k = 3, 4, \ldots, m-2;$$

$$b_{m-1} = f_{m-1} - \frac{\alpha k_1}{\Delta\tau^2}(u_m^* - 2u_{m-1}^* + u_{m-2}^*) + \frac{\alpha k_2}{\Delta\tau^4}(u_{m+1}^* - 4u_m^* - 6u_{m-1}^* - 4u_{m-2}^* + u_{m-3}^* - C_m\Delta\tau^2);$$

$$b_m = f_m - \frac{\alpha k_1}{\Delta\tau^2}(u_{m-1}^* - u_m^*) + \frac{\alpha k_2}{\Delta\tau^4}(u_{m-2}^* - 4u_{m-1}^* - 5u_m^* - 2u_{m+1}^* - 2C_m\Delta\tau^2);$$

and $C_m = du(\tau_m)/d\tau$ is specified beforehand.

We note that the value $u = f_0$ is the initial condition for the heat-conduction problem at the point at

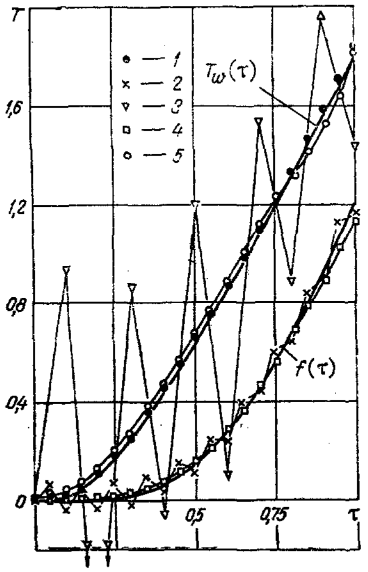

Fig. 1. Results of a solution of the model inverse problem through the use of exact, perturbed, and smoothed input data. Solid lines) exact numerical solution: 1) $T_W(\tau)$ for the case of exact input temperatures; 2) sawtooth perturbation of the input data, $\Delta f = \pm 0.05 f_{max}$, $j = 1, 2, \ldots, m$; 3) $T_W(\tau)$ for the case of perturbed data; 4) input function reconstructed by regularization method; 5) $T_W(\tau)$ for the case of smoothed data.

which the temperature is measured. In the course of the experiment this value can be measured quite accurately.

The best approximation is chosen on the basis of the "discrepancy principle" [6]: where

$$\rho(\alpha) \equiv \Big[\sum_{j=1}^{m} (u_j - f_j)^2\Big]^{\frac{1}{2}} = \delta, \quad \delta = \Big[\sum_{j=1}^{m} \sigma_j^2\Big]^{\frac{1}{2}}, \tag{23}$$

where σ_j is the mean square error of the input data at time j.

Condition (23) allows us to construct a method for automatically seeking the optimum approximation [5]. An algorithm of this type was discussed in [7]. If it is not possible to evaluate the measurement error δ during the experiments, use may be made of the quasi-optimum-parameter method of [8], which is based on the internal convergence of the regularized solutions.

In accordance with the algorithms given above, we compiled ALGOL-60 programs and carried out calculations for model problems on an M-220 computer. The results of one of these calculations are shown in Fig. 1. We assumed a hypothetical material whose thermal properties depend on the temperature in the following manner:

$$C(T) = 2 - \frac{1}{0.02T^2 + 0{,}7T + 1}, \quad \lambda(T) = 3 - \frac{30}{1.3T^2 + T + 15},$$
$$k(T) = \psi(T) = 0.$$

The initial temperature distribution is assumed constant and equal to zero. As the input data we used the temperature of the internal surface of a plate [at $x = X_2(\tau)$]. Our problem was to determine the temperature at $x = X_1(\tau)$. The boundaries of the plate moved according to the specified laws

$$X_1(\tau) = 0{,}1\tau(\tau + 1), \quad X_2(\tau) = 1 - 0{,}1\tau(\tau + 1).$$

The exact values of the input temperatures, $f(\tau)$, and the unknown function $T_W(\tau)$ were determined through a numerical solution of the heating problem for boundary-value problems of the second kind:

$$q_1(\tau) = 4 - \frac{156}{100\tau^2 + 39}, \quad q_2(\tau) = 0.$$

In these calculations we used an $n \times m = 50 \times 50$ difference grid. Calculation of one version without iterations in terms of the coefficients in the solution of the inverse problem required about 15 min. In this example the average value of the dimensionless integration time step was

$$\Delta \mathrm{Fo}_{av} = \left(\frac{\lambda}{C}\right)_{av} \frac{\Delta\tau}{(X_2 - X_1)^2_{av}} \approx 0.03.$$

To save computer time, we carried out the calculations without iterations with respect to the coefficients.

The function $T_W(\tau)$ which we found through the use of the exact data is stable. When perturbations are incorporated in the input temperatures, however, the calculation becomes unstable. Smoothing of the input data by the regularization method leads to a stable and quite accurate result, even if the errors are large (5% of f_{max}).

It should be noted that in the reconstruction of the function by the regularization method the results found through minimization of the first derivative alone ($k_2 = 0$) are not satisfactory. A satisfactorily smooth function is obtained if the functional in (21) is minimized with both the first and second derivatives with respect to the time ($k_1 = k_2 = 1$) taken into account.

To summarize, we can say that the smoothing of the input data permits an important extension of the range of applicability of direct methods for solving inverse heat-conduction problems.

We note, in conclusion, that the results found through the use of this algorithm to solve linear inverse problems show that the numerical solution on the basis of the implicit approximation scheme has "viscous" properties which are much better than those corresponding to direct algebraic methods for solving the integral equations. In the numerical approach, the minimum dimensionless time step, ΔFo_{cr}, is about 0.01.

NOTATION

C(T), specific heat per unit volume; λ(T), thermal conductivity; k(T), filtration coefficient; ψ(T), distributed heat source (or sink); T, temperature; x, ξ, coordinate; h, coordinate integration step; t, τ, time; $\Delta\tau$, time integration step; D_1, D_2, range of integration of the heat-conduction equation; φ(x), initial temperature distribution; q, heat flux; X, coordinate of boundary of object; f, input data; δ, error in input temperatures; A, operator; u, function; f_δ, input data specified with certain errors; $T_w(\tau)$, temperature of the internal surface; Fo, Fourier number; ΔFo, increment in Fourier number; τ_m, boundary value at the right-hand limit of the time interval; $\|\cdot\|$, norm; α, regularization parameter.

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